AIAA JOURNAL
Vol. 33, No. 10, October 1995

Technical Notes

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Analysis of the Baldwin–Barth and Spalart–Allmaras One-Equation Turbulence Models

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I. Introduction

THERE has been renewed interest in one-equation turbulence models recently, as exemplified by the work of Baldwin and Barth[1] and Spalart and Allmaras.[2] These models have been formulated after more than two decades of silence from the research community. Continuing advances in computer resources have rekindled interest in turbulence models of all types. Since neither algebraic nor two-equation formulations have provided completely satisfactory results for general fluid mechanics applications, the motivation for re-examining one-equation models is clear.

As a brief overview, the earliest one-equation model was formulated by Prandtl.[3] The model was based on the turbulence kinetic energy (TKE) equation. The eddy viscosity ν_T is computed as the product of $\sqrt{k}$, where k is turbulence kinetic energy, and a specified turbulence length scale. Emmons[4] and Glushko[5] also worked with models similar to the one proposed by Prandtl. Bradshaw et al.[6] presented an interesting variant of the TKE model that avoids introducing gradient-diffusion closure approximations. Their model performed well for incompressible boundary layers. Finally, Nee and Kovasznay[7] proposed a simplified one-equation model in which the eddy viscosity is computed directly.

The Baldwin–Barth[1] and Spalart–Allmaras[2] models are obviously generalizations of the Nee–Kovasznay[7] formulation. Here, we have analyzed the Baldwin–Barth and Spalart–Allmaras models. Although the original intent had been to include the Nee–Kovasznay model in the analysis, we found that formulation of the model was never actually completed. This precluded its inclusion.

In general, turbulence models are fine tuned for specific flows. Testing is often confined to these cases. The hope is that such models are sufficiently general to apply to similar flows and, perhaps, to an even wider range. However, there is inherent risk in using any turbulence model, namely, it might be applied to a flow it was never meant to solve. Thus, it is helpful to establish a model's range of applicability, using objective analytical methods.

At a minimum, every turbulence model should be shown to be capable of handling certain basic flows such as the equilibrium turbulent boundary layer, including effects of pressure gradient. This is true since there are a limited number of practical engineering flows that have constant pressure, and those that do are often trivial.

In this Note, we use perturbation methods to analyze Baldwin–Barth and Spalart–Allmaras model predictions for the defect layer. Additionally, we have exercised the models in 16 boundary-layer computations. Results of the perturbation analysis are completely consistent with the boundary-layer computations. Since two different numerical integration schemes have been used to obtain the same overall conclusions, confidence in the results is very high.

Received Oct. 11, 1994; revision received Jan. 10, 1995; accepted for publication Jan. 13, 1995.

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II. Models Selected

As already noted, the one-equation models studied were two versions of the Baldwin–Barth model (denoted B1 and B2) and the Spalart–Allmaras (denoted SA) model. B1 refers to the Baldwin–Barth model using its originally specified (unmodified) freestream value of the eddy viscosity ($\nu_T \rightarrow 0$), and B2 refers to the Baldwin–Barth model using a value for the freestream eddy viscosity ($\nu_T \rightarrow 30\nu$) that yields a satisfactory (optimized) solution for the constant pressure boundary layer.

A. Mean Flow Equations

The Reynolds-averaged equations for conservation of mass and momentum are the same for both models. The equations for the incompressible boundary layers are as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{\partial}{\partial y}\left[(\nu + \nu_T)\frac{\partial u}{\partial y}\right] \tag{2}$$

where x and y are streamwise and normal distances, u and v are streamwise and normal velocity components, and p and ρ are fluid pressure and density, whereas ν and ν_T are molecular and eddy viscosity, respectively.

B. Model Equations

1. Baldwin–Barth Model Equations

For the Baldwin–Barth model the kinematic eddy viscosity ν_T is given by

$$\nu_T = C_\mu \nu \tilde{R}_T D_1 D_2 \tag{3}$$

where C_μ is a closure coefficient, ν is the molecular viscosity, and $\tilde{R}_T$ is the turbulence Reynolds number.

The quantity $\tilde{R}_T$ satisfies the following equation:

$$u\frac{\partial \tilde{R}_T}{\partial x} + v\frac{\partial \tilde{R}_T}{\partial y} = (C_{\varepsilon 2} f_2 - C_{\varepsilon 1})\sqrt{\frac{\tilde{R}_T P}{\nu}} + \left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial^2 \tilde{R}_T}{\partial y^2} - \frac{1}{\sigma_\varepsilon}\frac{\partial \nu_T}{\partial y}\frac{\partial \tilde{R}_T}{\partial y} \tag{4}$$

The various closure coefficients and closure functions for this model are

$$C_{\varepsilon 1} = 1.2 \quad C_{\varepsilon 2} = 2.0 \quad C_\mu = 0.09 \quad A_0^+ = 26 \quad A_2^+ = 10 \tag{5}$$

$$1/\sigma_\varepsilon = (C_{\varepsilon 2} - C_{\varepsilon 1})\left(\sqrt{C_\mu}/\kappa^2\right), \qquad \kappa = 0.41 \tag{6}$$

$$P = \nu_T\left(\frac{\partial u}{\partial y}\right)^2 \tag{7}$$

$$D_1 = 1 - e^{-(y^+/A_0^+)} \qquad D_2 = 1 - e^{-(y^+/A_2^+)} \tag{8}$$

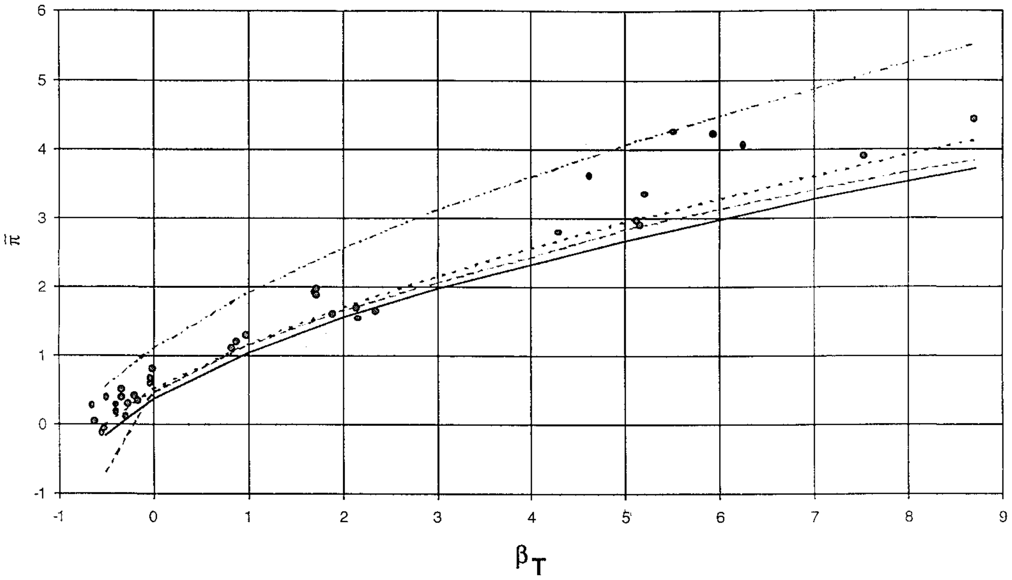

Fig. 1 Wake strength parameter as a function of pressure gradient: - - - - -, k–ω; ——, SA·······, B1; · · · · ·, B2; ○, experiment.

$$f_2 = \frac{C_{\varepsilon 1}}{C_{\varepsilon 2}} + \left[1 - \left(\frac{C_{\varepsilon 1}}{C_{\varepsilon 2}}\right)\right]\left[\frac{1}{\kappa y^+} + D_1 D_2\right]$$
$$\times \left[\sqrt{D_1 D_2} + \frac{y^+}{\sqrt{D_1 D_2}}\left(\frac{D_2}{A_0^+}e^{-(y^+/A_0^+)} + \frac{D_1}{A_2^+}e^{-(y^+/A_2^+)}\right)\right] \quad (9)$$

2. *Spalart–Allmaras Model Equations*

For the Spalart–Allmaras model, the kinematic eddy viscosity ν_T is given by

$$\nu_T = \tilde{\nu} f_{v1} \quad (10)$$

where f_{v1} is a closure coefficient and the quantity $\tilde{\nu}$ satisfies

$$u\frac{\partial \tilde{\nu}}{\partial x} + v\frac{\partial \tilde{\nu}}{\partial y} = C_{b1}\tilde{\nu}\tilde{S} - C_{w1} f_w \left(\frac{\tilde{\nu}}{y}\right)^2$$
$$+ \frac{1}{\sigma}\frac{\partial}{\partial y}\left[(\nu + \tilde{\nu})\frac{\partial \tilde{\nu}}{\partial y}\right] + \frac{C_{b2}}{\sigma}\left(\frac{\partial \tilde{\nu}}{\partial y}\right)^2 \quad (11)$$

The closure coefficients and closure functions appearing in Eq. (11) are

$$C_{b1} = 0.1355 \qquad C_{b2} = 0.622 \qquad C_{v1} = 7.1 \qquad \sigma = 2/3 \quad (12)$$

$$C_{w1} = \frac{C_{b1}}{\kappa^2} + \frac{(1 + C_{b2})}{\sigma} \qquad C_{w2} = 0.3 \qquad C_{w3} = 2.0$$
$$\kappa = 0.41 \quad (13)$$

$$f_{v1} = \frac{\chi^3}{\left(\chi^3 - C_{v1}^3\right)}; \qquad f_{v2} = 1 - \frac{\chi}{(1 + \chi f_{v1})}$$
$$f_w = g\left[\frac{\left(1 + C_{w3}^6\right)}{\left(g^6 + C_{w3}^6\right)}\right]^{\frac{1}{6}}; \qquad \chi = \frac{\tilde{\nu}}{\nu} \quad (14)$$

$$g = r + C_{w2}(r^6 - r); \qquad r = \left[\frac{\tilde{\nu}}{(\tilde{S}\kappa^2 y^2)}\right]$$
$$\tilde{S} = \left|\frac{\partial u}{\partial y}\right| + \left[\frac{\tilde{\nu}}{(\kappa^2 y^2)} f_{v2}\right] \quad (15)$$

C. Methods

During the analysis phase of this study, we performed a perturbation analysis of the defect layer, which is ideally suited to determine turbulence model behavior for boundary layers with pressure gradients. This is because a model's ability to represent the effects of pressure gradients can be quantified with a minimum amount of analysis, thereby enabling us to make an informed decision on the applicability of the model. This approach is strictly valid only for equilibrium boundary layers (i.e., flows where the so-called equilibrium parameter β_T is constant), which was the case during this study (Fig. 1). This does not really limit the applicability of perturbation analysis, however, as Wilcox[8] has established that the perturbation solution holds even for the nonequilibrium case, in a quasiequilibrium sense.

We performed the perturbation analysis as described in Wilcox[8] (details omitted for brevity) on the steady-state model equations for the Baldwin–Barth model [Eq. (4)], and the Spalart–Allmaras model [Eq. (11)]. This leads to the following transform equations. These equations were then coded into the program DEFECT[8] (for the defect-layer solution).

Baldwin–Barth model:

$$\frac{N_0}{\sigma_\varepsilon}\left(\frac{\mathrm{d}^2 N_0}{\mathrm{d}\eta^2}\right) - \frac{1}{\sigma_\varepsilon}\left(\frac{\mathrm{d}N_0}{\mathrm{d}\eta}\right)^2 + (1 + \beta_T)\eta\frac{\mathrm{d}N_0}{\mathrm{d}\eta}$$
$$- (1 + 2\beta_T)N_0 - \frac{\kappa^2}{\sigma_\varepsilon}N_0\frac{\mathrm{d}U_1}{\mathrm{d}\eta} = 0 \quad (16)$$

Spalart–Allmaras model:

$$\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(N_0\frac{\mathrm{d}N_0}{\mathrm{d}\eta}\right) - \frac{C_{b2}}{\sigma}\left(\frac{\mathrm{d}N_0}{\mathrm{d}\eta}\right)^2$$
$$+ (1 + \beta_T)\eta N_0 - C_{w1} f_w\left(\frac{N_0}{\eta}\right)^2 - C_{w1}N_0\frac{\mathrm{d}U_1}{\mathrm{d}\eta} = 0 \quad (17)$$

The quantity β_T is the equilibrium parameter defined by

$$\beta_T = \frac{\delta^*}{\tau_w}\frac{\mathrm{d}p}{\mathrm{d}x} \quad (18)$$

where δ^* is displacement thickness, τ_w is surface shear stress, and $\mathrm{d}p/\mathrm{d}x$ is pressure gradient.

Additionally, we have coded the model equations [Eqs. (3–15)] into program EDDYBL,[8] which can be used for general boundary-layer computations. The results have been compared to those from the k–ω model,[9] which was used as a yardstick to test the validity of the results. Before performing our analysis, however, each program has been tested for grid sensitivity using Richardson extrapolation. Both DEFECT and EDDYBL were found to be within 1% for the Spalart–Allmaras model, and within 4% for the two versions of the Baldwin–Barth model. We also note that, for both the Spalart–Allmaras and the Baldwin–Barth models, it was much harder to obtain converged solutions. With program DEFECT, for example, these models would run at Courant–Friedrichs–Lewy (CFL) numbers only 1/3–1/2 of what is possible with the k–ω model. This is somewhat of a surprise, since the models' formulators have implied otherwise.

Results and Discussion

As an overview, program DEFECT has been used to compute the wake-strength parameter $\tilde{\pi}$ for various values of the equilibrium parameter β_T. Program EDDYBL has been used to compute the skin friction coefficient c_f as a function of distance along the surface. For this study, we analyzed the same 16 cases as Wilcox.[9]

Figure 1 highlights the fact that according to our perturbation analysis, the Spalart–Allmaras model's performance is comparable to that of the k–ω model. By contrast, virtually any kind of agreement can be forced with the Baldwin–Barth model because of its sensitivity to freestream conditions. Perturbation analysis demonstrates that the Spalart–Allmaras model's performance is expected to be far superior to the Baldwin–Barth model, although not quite as good as the k–ω model.

On the one hand, Fig. 1 shows that the unmodified B1 model responds too much and too early to adverse pressure gradients, which in turn points to a tendency to predict separation earlier than has been observed experimentally. On the other hand, the optimized B2 model's results seem to be comparable to those of the Spalart–Allmaras and k–ω models. However, this is somewhat misleading, since it would imply that there might be some value to pursuing the optimized B2 model. In fact, the unmodified B1 model is not the asymptote $N_0(\infty) \to 0$. Specifically, the difference between the results of the optimized B2 model and the unmodified B1 model corresponds to reducing $N_0(\infty)$ from 0.002 to 0.0002, respectively. In fact, the behavior due to further reduction in the value of ν_T was also studied. We found that numerical difficulties increase dramatically for $N_0(\infty)$ smaller than 0.0001. The deviation from the optimized B2 model's behavior was so severe as to make further pursuit of this line of inquiry pointless.

The numerical difficulties encountered with the Baldwin–Barth model are caused by the kinematic eddy viscosity ν_T shutting off

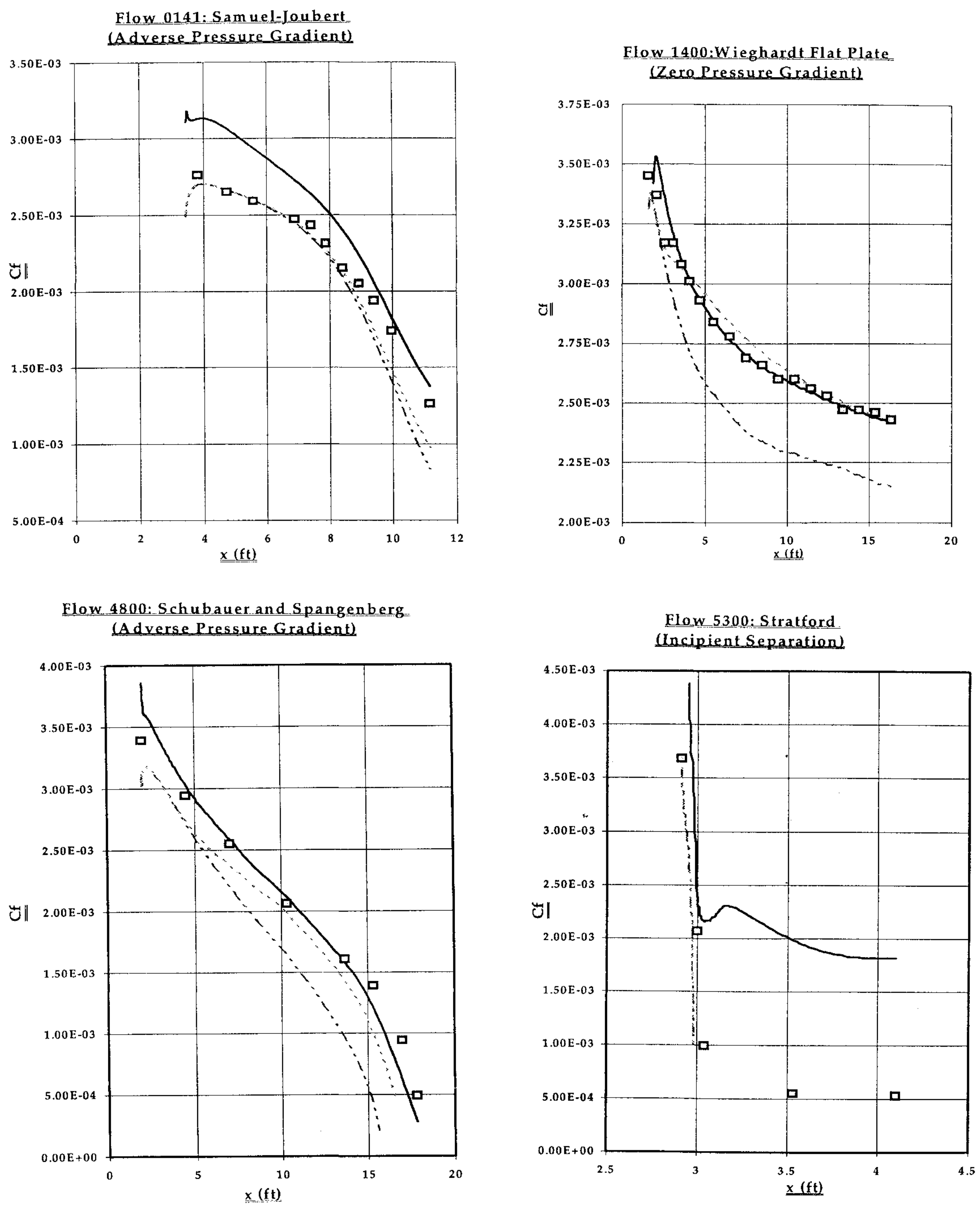

Fig. 2 Skin friction for equilibrium boundary layers: - - - -, B1; · · · · · ·, B2; ——, SA; □, experiment.

abruptly before reaching the edge of the boundary layer. This is physically unrealistic since ν_T has a nonzero value in the boundary layer and asymptotically approaches zero in the freestream. Also, this tendency makes it extremely difficult to integrate the velocity profile through the boundary layer. Thus, it is necessary to specify a nonzero cutoff value in the code for the freestream ν_T to avoid numerical difficulties and to obtain physically meaningful results. Thus, in effect, the freestream value of ν_T serves as an additional adjustable parameter for the model.

Turning to the computations with EDDYBL, results are completely consistent with the perturbation analysis. Results for only 4 of the 16 flows computed are shown in Fig. 2, but they are very representative of all 16 cases. Using the same criterion as Wilcox,[9] the average difference between the computed and measured c_f for the 16 cases is 6% for the k–ω model, 14% for the Spalart–Allmaras model, 21% for the optimized Baldwin–Barth model, and 27% for the unmodified Baldwin–Barth model.

Further, as shown in Fig. 2, for the unmodified B1 model computed skin friction c_f is 11% lower than measured for the constant pressure case (flow 1400). The freestream value for ν_T was selected to be 30ν for the optimized B2 model in order to force agreement with measured c_f for the flat plate case. [Note that using $\nu_T = 30\nu$, corresponds to $N_0 = (\nu_T/Ue\delta^*) = (30/Re\delta^*) \sim 0.001$, for the flows considered in Fig. 2.] Nevertheless, using this value of ν_T yields modest improvements in predictions with other cases, although premature separation still cannot be avoided. Consistent with the perturbation analysis, separation occurs earlier for smaller values of ν_T in the freestream. The Baldwin–Barth models actually predict separation for flows 4800 and 5300, whereas measurements indicate these flows remain attached. By contrast, Spallart–Allmaras predictions are much closer to measurements for all 16 cases. However, as shown in Fig. 2, the Spallart–Allmaras c_f is more than 200% higher than measured for flow 5300 (Stratford's incipient separation case). By contrast, for the k–ω model, computed c_f is only 19% above measured values for flow 5300.

As a final comment, since the two programs, i.e., DEFECT and EDDYBL, use different numerical integration schemes, we can reasonably have high confidence in the results.

Acknowledgments

We would like to express our gratitude to D. C. Wilcox (President, DCW Industries, Inc.) for suggesting this problem for study as well as for his invaluable guidance and comments throughout the course of the present study. We would also like to acknowledge the guidance and support of our thesis advisors at the University of Southern California, Los Angeles, L. C. Wellford and E. P. Muntz, during this study.

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Near-Wall Integration of Reynolds Stress Turbulence Closures with No Wall Damping

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Introduction

THE reliable computation of complex wall-bounded turbulent flows requires the direct integration of Reynolds stress closures to a solid boundary with the no-slip condition applied. Laws of the wall boundary conditions do not formally apply to complex turbulent flows with separation or with body force effects arising from streamline curvature or a system rotation. To perform near-wall integrations, traditional full Reynolds stress closures—as well as most existing two-equation models—require the introduction of a variety of ad hoc wall damping and wall-reflection terms that depend on the distance from the wall as well as the unit normal to the wall. This makes it virtually impossible to apply full Reynolds stress closures to wall-bounded turbulent flows within complex geometries where the local wall distance or unit wall normal may not be uniquely defined. Whereas there do exist two-equation models of the K–ε type that have wall damping functions that only depend on the turbulence Reynolds number[1] they, too, have problems in complex turbulent flows. These problems arise from the fact that the damping functions are completely ad hoc; since they contain virtually no turbulence physics and are calibrated based on the equilibrium turbulent boundary layer, they usually break down when applied to more complex turbulent flows.

The serious difficulties with wall damping functions just outlined explain the popularity of the K–ω model of Wilcox[2] that, along with its variations, has constituted the only existing two-equation turbulence model that can be integrated directly to a solid boundary with no wall damping. Despite this positive feature, however, there are other difficulties with the K–ω model. For example, its use of the inverse time scale $\omega(\equiv \varepsilon/K)$ renders the model to be overly sensitive to the freestream boundary conditions in external flows. Furthermore, since it is based on an isotropic eddy viscosity, the K–ω model suffers from the same deficiencies as the standard K–ε model in the description of more complex turbulent flows involving streamline curvature or a system rotation.[3] This forms the motivation for the current study: to present a two-equation model of the K–ε type that accounts for more turbulence physics and can be integrated directly to a solid boundary with no ad hoc wall damping functions in the representation for the Reynolds stress tensor.

The model that will be considered is the explicit algebraic stress model of Gatski and Speziale.[4] This constitutes a two-equation model with an anisotropic eddy viscosity that is systematically derived from the SSG second-order closure model[5] via the algebraic stress approximation for equilibrium turbulent flows. The SSG model has two notable advantages: 1) it is the generic form of the standard hierarchy of second-order closures, optimally calibrated for two-dimensional mean turbulent flows near equilibrium and 2) it requires no ad hoc wall reflection terms in order to yield accurate predictions for the logarithmic region of an equilibrium turbulent

Received April 24, 1995; revision received May 30, 1995; accepted for publication June 1, 1995.

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